Wallis et Stirling (mini chapitre)

Intégrales de Wallis :

$$I_n = \int_0^{\pi/2} \sin^n t \, dt = \int_0^{\pi/2} \cos^n t \, dt \quad , \quad I_0 = \pi/2 \, , \, I_1 = 1$$

1) $I_n \to 0$ Soit $\varepsilon > 0$, on coupe à $\pi/2 - \varepsilon$

$$0 \leq I_n = \int_0^{\pi/2-\varepsilon} \sin^n t \, dt + \int_{\pi/2-\varepsilon}^{\pi/2} \sin^n t \, dt \leq \int_0^{\pi/2-\varepsilon} \sin^n t \, dt + \varepsilon$$

$$\leq \left(\sin(\pi/2 - \varepsilon)\right)^n \pi/2 + \varepsilon$$

$$\exists n_\varepsilon \in \mathbb{N} \; \forall n \geq n_\varepsilon \quad 0 \leq \sin^n(\pi/2 - \varepsilon) \leq \varepsilon$$

Delà $\forall n \geq n_\varepsilon$, $0 \leq I_n \leq 2\varepsilon$

2) calculs.

Récurrence $n \geq 2$
$$\int_0^{\pi/2} \sin^n t \, dt = \int_0^{\pi/2} \sin^{n-2} t \times (1 - \cos^2 t) \, dt$$

$$= I_{n-2} - \int_0^{\pi/2} \underbrace{(\sin^{n-2} t \cos t)}_{u'} \underbrace{\cos t}_{v}$$

$$= I_{n-2} - \underbrace{[\ldots]_0^{\pi/2}}_{0} - \int_0^{\pi/2} \frac{\sin^{n-1} t}{n-1} \sin t \, dt$$

$$= I_{n-2} - \frac{I_n}{n-1} \quad \text{donc}$$

$$I_n \left(1 + \frac{1}{n-1}\right) = I_{n-2} \quad \text{ie } I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{2p} = \frac{2p-1}{2p} I_{2p-2} \quad \left| \; I_{2p} = \frac{(2p-1)\cdots 1}{2p \cdots 2} \pi/2 = \frac{(2p)!}{2^{2p}(p!)^2} \pi/2 \right.$$

$$I_{2p+1} = \frac{2p}{2p+1} I_{2p-1} \quad \Bigg| \quad I_{2p+1} = \frac{2p(2p-2)\cdots 2 \times 1}{(2p+1)(2p-1)\cdots 3} = \frac{2^{2p}(p!)^2}{(2p+1)!}$$

Equivalent i) $n I_n I_{n-1} = cste = \pi/2$

Vérifions $(n+1) I_{n+1} I_n = (n+1)\left(\frac{n}{n+1} I_{n-1}\right) I_n = n I_{n-1} I_n$

ii) $I_n < I_{n-1} < I_{n-2} = \underbrace{\frac{n-1}{n}}_{\to 1} I_n$ ← à corriger.

Ainsi $n I_n^2 \to \pi/2 \quad I_n \sim \sqrt{\frac{\pi}{2n}}$

<u>Stirling</u> :

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

1ère étape : On veut Mq $u_n = \frac{n!\, e^n}{n^{n+\frac{1}{2}}}$ converge (?) vers $C > 0$

Il s de Mq $v_n = \log u_n$ dans $\mathbb{R}$

ie que $\sum v_{n+1} - v_n$ CV

$v_{n+1} - v_n = 1 - \log\left(\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\right) = 1 - \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right)$

$= 1 - \left(n+\frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)$

$= -\frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^2}\right)$

donc $\sum v_{n+1} - v_n$ CV

alors v_n CV vers L donc $u_n \to e^L$

Calcul de C

On utilise Wallis (Wario! :ö) $I_{2p} \sim \sqrt{\frac{\pi}{4p}} \sim \frac{1}{2}\sqrt{\frac{\pi}{p}}$

Or $I_{2p} = \frac{(2p)!}{2^{2p}(p!)^2}\,\pi/2 \sim \frac{\left(\frac{2p}{e}\right)^{2p}\sqrt{2p}\,C\,\pi/2}{\cancel{2^{2p}}\left(\frac{p}{\cancel{e}}\right)^{2p} p\,C^2}$

$\sim \frac{1}{2}\sqrt{\frac{\pi}{p}}$

$= \frac{\pi}{2}\sqrt{\frac{2}{p}} \times \frac{1}{C}$

donc $\frac{\pi}{\sqrt{2}}\frac{1}{C} \sim \frac{1}{2}\sqrt{\pi}$

c.à.d $C \sim \sqrt{2\pi}$

d'où $C = \sqrt{2\pi}$

On en déduit que $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$